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Complex fiber bundle model for optimization of heterogeneous materials

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We propose a complex fiber bundle model for the optimization of heterogeneous materials, which consists of many simple bundles. We also present an exact and compact recursion relation for the failure probability of a simple fiber bundle model with local load sharing, which is more efficient than the ones reported previously. Using a ''renormalization method'' and the recursion relation developed for the simple bundle, we calculate the failure probabilities of the complex fiber bundle. When the total number of fibers is given, we find that there exists an optimum way to organize the complex bundle, in which one gets a stronger bundle than in other ways. [S1063-651X(96)13509-1]

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I. INTRODUCTION

For many years, fracture and failure of materials have drawn much attention of physicists. There have been many efforts to analyze the fracture and failure properties of heterogeneous materials with the use of random network models [1–4]. Among the many theoretical models for material failure the fiber bundle model has been studied extensively in recent years [5–10]. According to the load sharing rules, the fiber bundle models can be divided into two types: the equal load sharing (ELS) model and the local load sharing (LLS) model. In the ELS models, the load is shared equally by all surviving elements in the system. This is appropriate for loosely wound yarns. In the LLS models, the load previously carried by a failed element is shared by the surviving elements in the immediate vicinity. This kind of load sharing occurs in most materials under tensile loading, and is included in the random spring, electric, dielectric, and superconducting networks [11–13]. One aspect of the studies on the fiber bundle model concerns the strength of the bundle. The question often asked is, Under a given load, say σ, what is the probability that the fiber bundle fails? For the ELS model the failure probabilities can be calculated analytically [14,15]. The LLS model [16,17], however, is much more difficult to treat analytically. In the early studies on the LLS fiber bundle model, Harlow and Phoenix [18] developed a transition matrix method to calculate the failure probability. Lately some recursion relations were developed [19,20]. In our previous work, we developed an exact recursion relation [21] for calculating the failure probability in the LLS model. An interesting finding was that for a given external load σ, the failure probability as a function of the system size n has a well defined minimum at a certain value of n, say n_m (see Fig. 1). We are motivated to find an optimal way to arrange the fibers such that it gives a stronger bundle than other ways.

The fiber bundle we studied before is a one dimensional array of n individual fibers, whose thresholds are chosen randomly according to some distribution function $p(x)$, such that $p(x)dx$ is the probability that the threshold of a fiber is in $[x,x+dx]$. The fiber bundle may be called a simple bundle in the sense that its elements are merely individual fibers. In this paper, we study a complex bundle, which is organized with many simple bundles in the same way the simple bundle is organized with fibers. Now the simple bundles can be called sub-bundles, which are regarded as the elements of the complex bundle. We can then apply the recursion relation developed for the simple bundle to the complex bundle through some ''renormalization approach.'' When we deal with a simple bundle, the elements of it are individual fibers. At a higher level, when we deal with the complex bundle, each sub-bundle is now considered as an element. At the two different levels, the form of recursion relations remains the same. It is in this sense we use the term ''renormalization.'' In the earlier studies on the fiber bundle models, a chain-of-fiber bundles model was studied extensively [18,22–24]. In contrast to the chain of bundles, which is an organization of simple bundles in series, the complex fiber bundle is a parallel organization of simple bundles. Newman *et al.* [25] have proposed a hierarchically organized fiber bundle model, with equal load sharing. Their model has many levels of bundles, while our model in consideration only has two levels, the simple bundle and the complex bundle. It is straightforward to generalize our model to higher levels.

II. COMPACT RECURSION RELATION

For the simple bundle, we could calculate the failure probability through the exact recursion relation reported in our previous paper [21]. The recursion relation turned out to be more efficient than some approximate methods. However, in this paper, we do not intend to use that recursion relation for the calculations. Noticing the work by Duxbury and

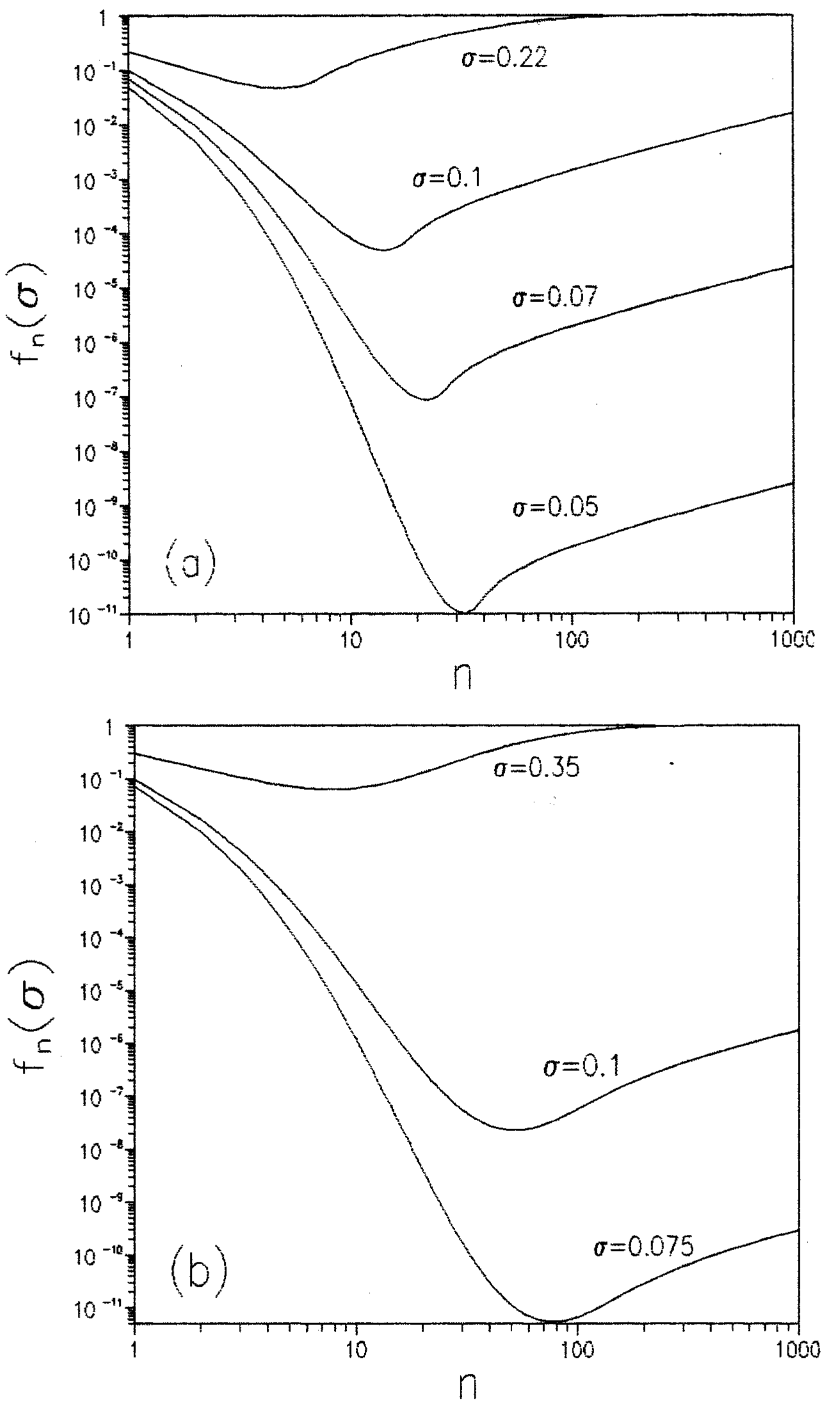

FIG. 1. The failure probability of a simple bundle, as a function of the number of fibers n, has a well defined minimum at n_m. As examples, (a) shows the result for the uniform threshold distribution $p(\sigma)=1$, where $\sigma\in[0,1]$; (b) shows the result for the Weibull threshold distribution $p(\sigma)$ such that $P(\sigma)=\int_0^\sigma p(x)dx=1-\exp[-\sigma^m]$ with $m=1$, where $\sigma\in[0,\infty)$.

Leath [26], here we develop another exact recursion relation, which will be shown to be more compact and more efficient than the previous one.

In this paper when we speak of the load on the system, we mean the total external force divided by the total number of fibers in the system. The local load sharing rules are defined such that a surviving element in the bundle (simple or complex) carries the load $(1+r/2)\sigma$, where σ is the load on the bundle, and r is the number of consecutive failed elements immediately adjacent to this surviving element counting on both sides. According to the load sharing rules, the probability $w_i(\sigma)$ that a fiber survives, when it has i failed fibers adjacent to it, is

$$w_i(\sigma)=1-\int_0^{(1+i/2)\sigma} p(x)dx. \tag{1}$$

When a given external load σ is applied to the fiber bundle, the probability $f_n(\sigma)$ that the fiber bundle fails is the interest of the studies. If all the fibers in the bundle fail, the bundle is said to have failed. If at least one fiber survives, the bundle is said to survive. When a load σ is applied to the bundle, the bundle can survive in 2^n-1 different possible ways, which are called survival configurations. If we denote a failed fiber with a 0, and a survival fiber with a 1, the survival configurations of the bundle of n fibers can be put in the form

$$(010100\ldots101)_n. \tag{2}$$

The subscript n outside the parentheses indicates the total number of fibers in the bundle, while the 1's and 0's in the parentheses show the status of corresponding fibers in the bundle. The probability of each survival configuration can be written out in explicit form. For example, the probability of the survival configuration $(0010010)_7$ is

$$s(0010010)_7 = f_2 w_4 f_2 w_3 f_1. \tag{3}$$

In the above expression, f_2 means $f_2(\sigma)$, which is the failure probability of a bundle of two fibers when a load σ is applied. We have suppressed the symbol σ.

Each of the 2^n-1 survival configurations is independent, so the survival probability of the bundle $s_n(\sigma)$ is obviously

$$s_n(\sigma) = \sum s(\text{ configuration})_n. \tag{4}$$

Then the failure probability of the bundle is just

$$f_n(\sigma) = 1 - s_n(\sigma). \tag{5}$$

However, the configuration-counting method, Eq. (4), is not applicable for large n, because the number of survival configurations scales with n as 2^n. It is in this situation that the various recursion relations were introduced.

For all the recursion relations developed previously [20,21,26], the key point is to classify the survival configurations properly, so that the survival probabilities of the configurations can be summed into groups, then the recursion relation is developed for the group probabilities. The major difference between the different recursion relations is the way in which the survival configurations are classified. We classify these 2^n-1 survival configurations into different groups in the following way. Let $s(n,i)$ be the set of survival configurations which have the form

$$(\ldots 1\overbrace{00\ldots0}^{i})_n, \tag{6}$$

where $0 \leqslant i \leqslant n-1$. In words, there are i consecutive 0's at the right ends of the configurations. So for given n, there are n groups of survival configurations. Each of the 2^n-1 survival configurations belongs uniquely to one of the n groups. For example, $(0101100)_7$ and $(1000100)_7$ belong to the set $s(7,2)$; $(111)_3$, $(001)_3$, and $(011)_3$ are included in the set $s(3,0)$. In addition, we can also let $s(n,n)$ denote the configuration $(000\cdots00)_n$, which is not a survival configuration but the failure configuration. So the probability of this configuration is virtually the failure probability f_n. Hereafter we also use $s(n,i)$ to denote the sum of the probabilities of the configurations in this set. So the survival probability of the bundle can be calculated through

$$s_n(\sigma) = \sum_{i=0}^{n-1} s(n,i). \tag{7}$$

Noticing that $s(n,n) = f_n$ and $f_n + s_n = 1$, we also have the relation

$$\sum_{i=0}^{n} s(n,i) = 1. \tag{8}$$

For small n, the set probability can be easily calculated. When $n=1$, there are only two configurations, $(1)_1$ and $(0)_1$, which belong to sets $s(1,0)$ and $s(1,1)$, respectively. It is straightforward that

$$s(1,0) = w_0 = 1 - \int_0^\sigma p(x)\,dx \tag{9}$$

and

$$s(1,1) = f_1 = \int_0^\sigma p(x)\,dx = 1 - w_0. \tag{10}$$

For arbitrary n, the set probability $s(n,i)$ can be calculated through the following simple recursion relation:

$$s(n,i) = \sum_{j=0}^{n-i-1} s(n-i-1,j)\,w_{j+i} f_i, \quad 0 \leqslant i \leqslant n-1. \tag{11}$$

In deducing the recursion relation, Eq. (11), we have defined $s(0,0)=1$, i.e., $f_0=1$. Remembering that $s(i,i)=f_i$, we can obtain the failure probability f_n for any n by using Eqs. (5), (7), and (11).

We note here that in Ref. [18] Harlow and Phoenix developed a Markov recursion for some conditional survival probabilities of the fiber bundle, which is in the form

$$\underline{Q}^{[k]} Q_n^{[k]} = Q_{n+1}^{[k]}, \tag{12}$$

where $Q_n^{[k]}$ is a column vector with 2^k-1 components, and $\underline{Q}^{[k]}$ is a 2^k-1 by 2^k-1 matrix. The first component of the vector $Q_{n+1}^{[k]}$, denoted by $Q_n^{[k]}(\sigma)$, is the probability that no sequence of k or more broken fiber elements will occur anywhere in a bundle of n fibers under load σ. For $k=n$, $Q_n^{[k]}(\sigma)$ is nothing but the survival probability of a bundle of n under load σ, which in this paper is denoted by $s_n(\sigma)$. For $k=n$, if one writes out Eq. (12) in summation form, the survival probability of the bundle $Q_n^{[n]}(\sigma)$ can be expressed as a sum of 2^n-1 terms, each of which is a product of a component of the vector $Q_n^{[n]}$ with the corresponding component of the matrix $\underline{Q}^{[n]}$. Equation (12) is a Markov recursion in the sense that $Q_{n+1}^{[k]}$ can be calculated from $Q_n^{[k]}$. The recursion relation, Eq. (11), we present in this paper, however, is not a Markov recursion, because $s(n,i)$ cannot be calculated merely from $s(n-1,j)$'s; we need to use the survival probabilities for all sizes less than n.

The recursion relation we present in this paper is more compact and efficient than the one reported in Ref. [21]. The new recursion relation, Eq. (11), contains only one expression, while the previous one contains three expressions [see Eq. (3.5) in Ref. [21]]. Using this recursion relation to calculate the survival probability $s_n(\sigma)$, the number of terms to be added is $n^2/2+n/2$, while for the previous one it is n^2-n+1. In computations, this new one requires only about 50% of the computer memory required by the previous one, and spends less CPU time than the latter. For example, in the computation for $f_n(0.14)$, from $n=1$ through $n=1000$, the CPU expenditure (Sun SPARC Station 1+) for the new recursion relation is about 260 sec, while for the previous one it is about 1800 sec. A subtle difference between the two recursion relations is that the new one does not involve division while the previous one did. Indeed, it is easier and faster and even sometimes more accurate for computers to do multiplication than to do division. For this reason we might expect that under some circumstances the new recursion relation be more accurate than the previous one. In most cases, the two recursion relations give exactly the same results for the failure probabilities.

We now can use the recursion relation (11) to calculate the survival probabilities and then the failure probabilities $f_n(\sigma)$'s of the LLS fiber bundles. For some given σ's, the plots of $f_n(\sigma)$ versus n are shown in Fig. 1. We see that the failure probability $f_n(\sigma)$, as a function of n, has a well defined minimum at a certain value of n, say n_m. It is the existence of this minimum that motivated us to consider constructing a complex fiber bundle and optimizing it. We also see from this figure that the position of n_m is σ dependent. Generally speaking, n_m becomes larger as σ is decreased, as shown in Fig. 2. A similar result about the behaviors of $f_n(\sigma)$ and n_m has been obtained by Leath and Duxbury [20], but their results were based on approximate calculations.

III. RENORMALIZATION APPROACH TO THE COMPLEX BUNDLE

Now we turn to the complex bundle and put this question: When a load σ is applied to a complex bundle of $n\times N$ fibers, what is the probability $F_N^{(n)}(\sigma)$ that the complex bundle fails? Here n is the number of fibers in each sub-bundle, and N is the number of sub-bundles in the complex bundle. So the total number of fibers in the system is nN. Remembering that the elements of the complex bundle are sub-bundles, we can now treat the complex model as a simple one.

The complex bundle also has 2^N-1 surviving configurations that can also be written in the form $(1000\cdots010)_N$. Letting $W_i^{(n)}$ be the probability that a sub-bundle of n fibers survives when it has i broken sub-bundles adjacent to it (counting on both sides), we have

$$W_i^{(n)}(\sigma)=1-f_n\left(\left(1+\frac{i}{2}\right)\sigma\right), \tag{13}$$

where $f_n(\sigma_s)$ is the probability that a sub-bundle fails when a load σ_s is applied on the sub-bundle, which can be calculated through Eq. (5), Eq. (7), and Eq. (11). The classification of the surviving configurations of the complex bundle is the same as for a simple bundle except that the elements are not fibers but sub-bundles. $S^{(n)}(N,i)$ is the set of surviving configurations of the complex bundle that have the form

$$(\ldots 1\overbrace{00\ldots 0}^{i})_N.$$

Then the recursion relation for $S^{(n)}(N,i)$ is obtained by replacing $s(n,i)$, f_n, and w_i in Eq. (11) with $S^{(n)}(N,i)$, $F_N^{(n)}$, and W_i, respectively. Then the failure probability of the complex bundle can be calculated by using the recursion relation for $S^{(n)}(N,i)$, which is in the same form as Eq. (11).

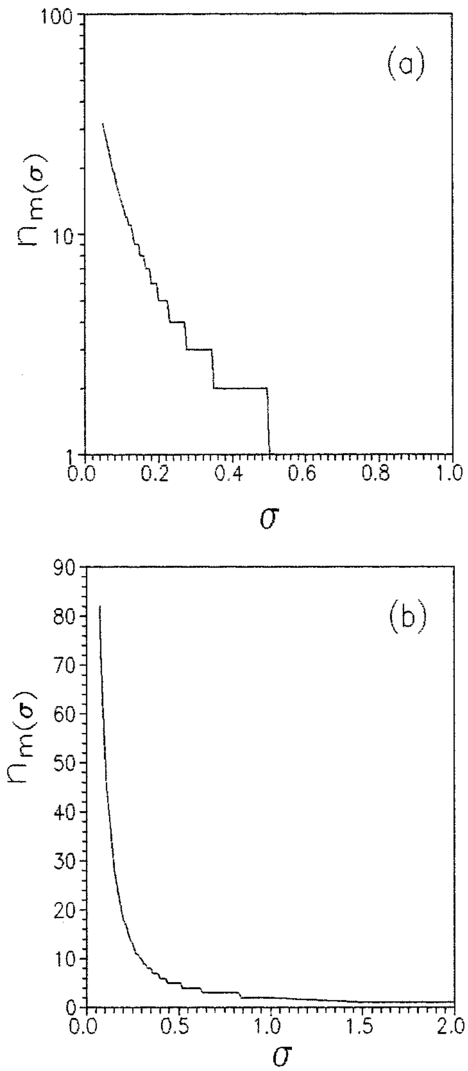

FIG. 2. The minima n_m, at which the failure probability assumes minimum value, depends on the load σ applied to the bundle. As $\sigma\to 0$, n_m may go to infinity, as shown in this figure. (a) For the uniform threshold distribution. (b) For the Weibull threshold distribution $P(\sigma)=1-\exp[-\sigma^m]$ with $m=1$.

IV. OPTIMIZATION OF THE COMPLEX BUNDLE

Suppose we have a total number Σ of fibers. The integer Σ can be factorized as $\Sigma=1\times\Sigma=n_1\times N_1=n_2\times N_2=\cdots=n_i\times N_i\cdots$. We can organize the Σ fibers as a simple fiber bundle or a complex fiber bundle which consists of N_i sub-bundles of n_i fibers. The question is, What is the best way to organize these Σ fibers such that it gives a stronger bundle than any other ways? In the case that the Σ fibers form a simple fiber bundle, when a load σ is applied to the system, the failure probability is just $f_\Sigma(\sigma)$. For a complex fiber bundle of the form $n_i\times N_i$, the failure probability is $F_{N_i}^{(n_i)}(\sigma)$. Our task is to find if there is an optimal way to construct the complex bundle, i.e., to find $N=n_o\times N_o$ such that

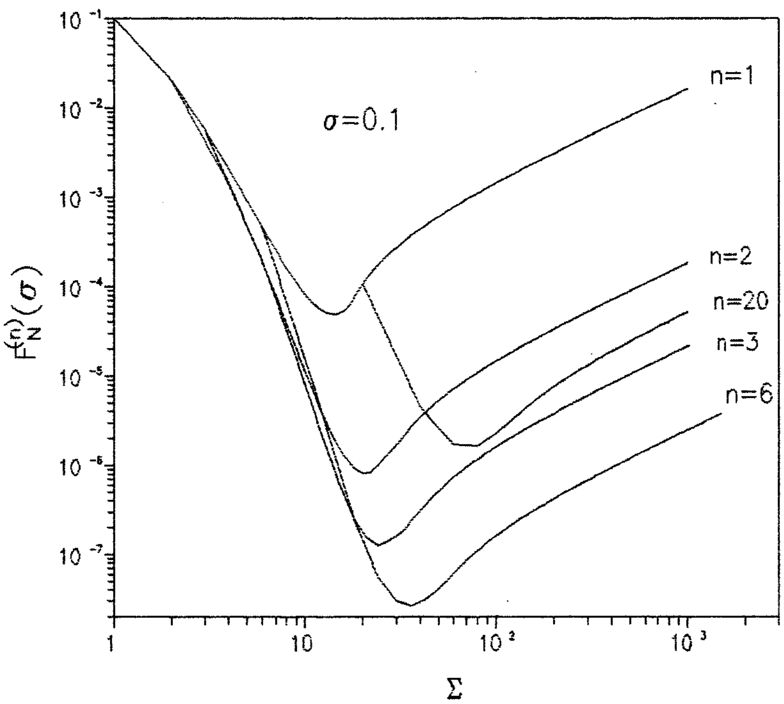

FIG. 3. The failure probabilities of the complex bundles for different n. When the load σ is given, the failure probability of the complex bundle, as a function of the total number of fibers Σ, has a similar behavior as that of a simple bundle. For a given Σ, different n gives different failure probabilities, among which the lowest one determines n_o. When $\sigma=0.1$, we get $n_o=6$. In this figure and the following figures, the threshold distribution is chosen to be the uniform one $p(\sigma)=1$, $\sigma\in[0,1]$.

$$F_{N_o}^{(n_o)}\leqslant F_{N_i}^{(n_i)}, \quad i\neq o. \tag{14}$$

The subscript o means optimal value.

In the following discussions, we take the uniform threshold distribution $p(\sigma)=1$, $\sigma\in[0,1]$ for examples. In Fig. 3, we present some of the results of the failure probabilities. The figure is a plot of $F_{N_i}^{(n_i)}$ versus the total number of fibers Σ. It is a natural outcome that each of the curves for different n has a point of intersection with the curve for $n=1$, i.e., the simple bundle, because when $N_i=1$ the complex bundle $n_i\times N_i$ is nothing but a simple bundle of n_i fibers. It is obvious that the lowest curve gives the optimal value n_o. When the load $\sigma=0.1$, we find that all the curves are above the curve for $n=6$. So $n_o=6$ is the optimal value of n for $\sigma=0.1$. We find that n_o is σ dependent, in other words, different σ values require different optimal n_o. In Table I, we list the failure probability of the complex bundle with a total number of fibers $\Sigma=2520$ but with different n and N. We see that when $\sigma=0.12$, the optimal value of n is $n_o=6$; when $\sigma=0.16$, however, the optimal value of n becomes $n_o=5$. So we should turn to some average quantities in order to find a unique n_o. The average strength of a complex fiber bundle can be calculated through

$$\overline{\sigma(n_i,N_i)}=\int \sigma dF_{N_i}^{(n_i)}(\sigma). \tag{15}$$

TABLE I. For $\Sigma=2520$, the failure probabilities of the complex bundles $F_N^{(n)}(\sigma)$ for the given load $\sigma=0.12$, and $\sigma=0.16$, and the average strength $\overline{\sigma(n,N)}$. The numbers marked with ⋆ are optimal value in each column.

	$F_N^{(n)}(0.12)$	$F_N^{(n)}(0.16)$	$\overline{\sigma(n,N)}$
$n=3$	0.003170420	0.307494	0.167071
4	0.001504400	0.242138	0.170267⋆
5	0.001066480	0.233358⋆	0.170191
6	0.000968673⋆	0.250782	0.168612
7	0.001049730	0.299856	0.166241
8	0.001181350	0.359475	0.163857
9	0.001430210	0.426635	0.161362

In Fig. 4 we show some results of the average strength of the complex bundle. The lines in this figure are calculated from Eq. (15), while the various symbols □, △, and ○ show the results from actual simulations of the complex bundle, which are in good agreement with the lines. In this figure, the

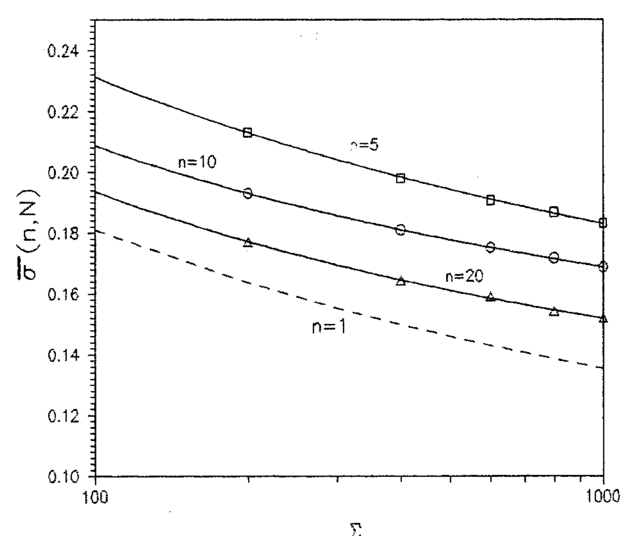

FIG. 4. The average strength of the complex bundles calculated from Eq. (15). The symbols □, △, and ○ represent the results obtained from actual simulations of 1000 samples. The dashed line in the figure is the average strength for a simple bundle. We see that the complex bundles are all stronger than a simple one.

TABLE II. The average strength of the complex bundle $\overline{\sigma(n,N)}$.

	$\Sigma=300$	600	1200	1800	2400
$n=3$	0.204041	0.189793	0.177881	0.171763	0.167731
4	0.205939	0.192181	0.180687	0.174794	0.170905
5	0.204086	0.191127	0.180234	0.174566	0.170808
6	0.201006	0.188472	0.178047	0.172709	0.169189

average strength for the simple bundle ($n=1$) is also shown, from which we see that all complex bundles are stronger than a simple bundle of the same total number of fibers Σ. We try to find a n_o such that for a given Σ

$$\overline{\sigma(n_o,N_o)} \geqslant \overline{\sigma(n_i,N_i)}, \qquad i \neq o. \tag{16}$$

The results about the average strength of the complex bundle are listed in Table II, which suggest that $n=4$ is the optimal value for the number of fibers in a sub-bundle when a complex bundle is to be constructed. The result that $n_o=4$ is true up to the total number of fibers $\Sigma=3000$.

We can also consider the failure probability density of the complex bundle $D_N^{(n)}(\sigma)$, which is defined as

$$D_N^{(n)}(\sigma)=\frac{\partial F_N^{(n)}(\sigma)}{\partial \sigma}. \tag{17}$$

As a function of σ, the failure probability density has a well defined maximum, which defines the most probable failure load σ_m, where the failure probability density assumes an extreme value. We have found for the simple bundle in our previous work that as the total number of fibers n goes to infinity, the most probable load σ_m actually coincides with the average strength of the bundle [21]. This result still holds for the complex bundle. Now we can optimize the complex bundle by the determination of n_o through the position of σ_m. In Fig. 5, we present the results for some complex bundles, which have the same total number of fibers $\Sigma=1000$, but with different n and N. The positions of σ_ms vary from one complex bundle to another. We find that the largest σ_m is given by the curve for $n=4$, indicating that $n=4$ is the optimal value n_o. The result for the optimal n obtained from the failure probability density is consistent with the result based on the average strength.

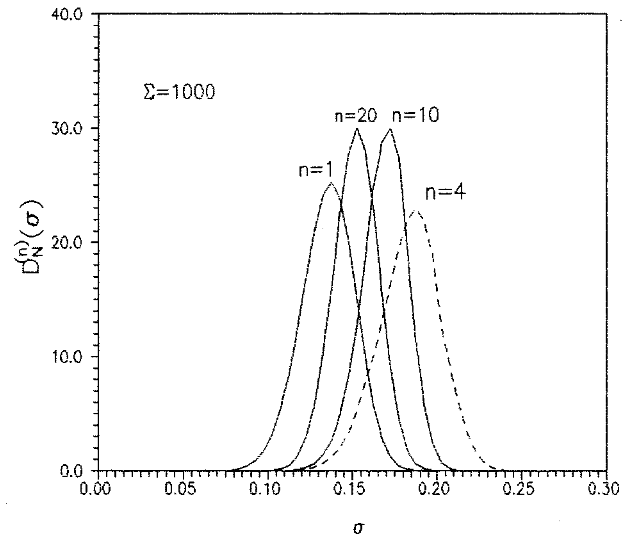

FIG. 5. The failure probability density $D_{N_i}^{(n_i)}(\sigma)$ for some complex bundles with total number of fiber $\Sigma=1000$. The most probable load σ_m is dependent on the number of fibers n in a sub-bundle. When $n=4$, σ_m assumes its optimal value.

ACKNOWLEDGMENTS

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[1] *Statistical Models for the Fracture of Disordered Media*, edited by H.J. Herrmann and S. Roux (North-Holland, Amsterdam, 1990).
[2] P.M. Duxbury, P.D. Beale, and P.L. Leath, Phys. Rev. Lett. **57**, 1052 (1986).
[3] P.D. Beale and D.J. Srolovitz, Phys. Rev. B **37**, 5500 (1988).
[4] P.L. Leath and W. Tang, Phys. Rev. B **37**, 6485 (1989).
[5] D. Sornette and S. Redner, J. Phys. A **22**, L619 (1989).
[6] D. Sornette, J. Phys. A **22**, L243 (1989).
[7] D.G. Harlow and S.L. Phoenix, J. Mech. Phys. Solids **39**, 173 (1991).
[8] Y.N. Lu and E.J. Ding, J. Phys. A **25**, L241 (1992).
[9] P.C. Hemmer and A. Hansen, J. Appl. Mech. **59**, 909 (1992).
[10] S.D. Zhang and E.J. Ding, Phys. Lett. A **193**, 425 (1994).
[11] W.A. Curtin and H. Scher, J. Mater. Res. **5**, 535 (1990).
[12] M. Sahimi and J.D. Goddard, Phys. Rev. B **33**, 7848 (1986).
[13] H.J. Herrmann, A. Hansen, and S. Roux, Phys. Rev. B **39**, 637 (1989).
[14] H.E. Daniels, Proc. R. Soc. London Ser. A **183**, 405 (1945).
[15] W. Lee, Phys. Rev. E **50**, 3797 (1994).
[16] R.L. Smith, Proc. R. Soc. London Ser. A **372**, 539 (1980).
[17] A. Hansen and P.C. Hemmer, Phys. Lett. A **184**, 394 (1994).
[18] D.G. Harlow and S.L. Phoenix, Int. J. Fracture **17**, 601 (1981).
[19] C.C. Kuo and S.L. Phoenix, J. Appl. Prob **24**, 137 (1987).
[20] P.L. Leath and P.M. Duxbury, Phys. Rev. B **49**, 14 905 (1994).
[21] S.D. Zhang and E.J. Ding, Phys. Rev. B **53**, 646 (1996).
[22] D.G. Harlow and S.L. Phoenix, J. Composite Mater. **12**, 195 (1978).
[23] D.G. Harlow and S.L. Phoenix, J. Composite Mater. **12**, 314 (1978).
[24] D.G. Harlow and S.L. Phoenix, Int. J. Fracture **17**, 347 (1981).
[25] W.I. Newman, A.M. Gabrielov, T.A. Durand, S.L. Phoenix, and D.L. Turcotte, Physica D **77**, 200 (1994).
[26] P.M. Duxbury and P.L. Leath, Phys. Rev. Lett. **72**, 2805 (1994).